REDIMAT
Journal of Research in Mathematics Education

Hipatia Press
www.hipatiapress.com

Instructions for authors, subscriptions and further details:

http://redimat.hipatiapress.com

Students' Reasoning on Multiplication in the Context of a Primary School Classroom

Odd Tore Kaufmann[1]

1) Østfold University College, Faculty of Education, Norway

Date of publication: February 24th, 2019
Edition period: February 2019-June 2019

To cite this article: Kaufmann, O.T. (2019). Students' reasoning on multiplication in the context of a primary school classroom. *REDIMAT – Journal of Research in Mathematics Education*, *8*(1), 6-29. doi: 10.4471/redimat.2019.2822

To link this article: http://dx.doi.org/10.4471/redimat.2019.2822

PLEASE SCROLL DOWN FOR ARTICLE

The terms and conditions of use are related to the Open Journal System and to Creative Commons Attribution License (CCAL).

REDIMAT, Vol. 8 No. 1 February 2019 pp. 6-29

Students’ Reasoning on Multiplication in the Context of a Primary School Classroom

Odd Tore Kaufmann
Østflod University College

(*Received: 09 September 2017; Accepted: 13 November 2018; Published: 24 February 2019)*

Abstract

This study examined third-grade students’ reasoning and discussions on multiplication when they initially encounter it in the classroom. The aim of the study was to analyse the data from teaching and learning multiplication in 24 classrooms and, thus make a contribution to the research and conceptualisations about students’ reasoning and strategy use in multiplication. The results revealed various features from previous findings, as well as new aspects which are helpful to characterise students’ multiplication reasoning. Findings revealed students employed general reasoning about the concept and different characteristics of multiplication. They broadened the context by going beyond the actual example and focusing on mathematical relationships. They placed strong emphasis on using addition when working with multiplication; this could cause tension in teacher-student discussions. The results are discussed in relation to previous studies of students’ multiplicative reasoning and implications for practice are elaborated upon.

Keywords: Multiplication, student’s strategies, reasoning.

2019 Hipatia Press
ISSN: 2014-3621
DOI: 10.4471/redimat.2019.2822

REDIMAT, Vol. 8 No. 1 Febrero 2019 pp. 6-29

El Razonamiento de los Estudiantes sobre Multiplicación en el Contexto de un Aula de Primaria

Odd Tore Kaufmann
Østflod University College

(*Recibido: 09 Septiembre 2017; Aceptado: 13 Noviembre 2018; Publicado: 24 Febrero 2019)*

Resumen

Este estudio examinó el razonamiento de los estudiantes de tercer curso y las discusiones sobre la multiplicación cuando inicialmente se encuentran en el aula. El objetivo del estudio fue analizar los datos sobre la enseñanza de la multiplicación que se llevaron a cabo en 24 aulas y, así, hacer una contribución a la investigación y conceptualización sobre el razonamiento de los estudiantes y el uso de la estrategia en la multiplicación. Los resultados revelan varias características de las contribuciones anteriores, así como nuevos aspectos que son útiles para caracterizar el razonamiento multiplicativo de los estudiantes. Las contribuciones revelan que los estudiantes emplean un razonamiento general sobre el concepto y las diferentes características de la multiplicación. Amplían el contexto yendo más allá del ejemplo real y centrándose en las relaciones matemáticas. Ponen énfasis en el uso de la suma cuando trabajan la multiplicación. Esto podría causar tensión en las discusiones entre profesores y alumnos. Los resultados se discuten en relación con estudios previos del razonamiento multiplicativo de los estudiantes y se explican las implicaciones para la práctica.

Palabras clave: Multiplicación, estrategias del alumnado, razonamiento.

2019 Hipatia Press
ISSN: 2014-3621
DOI: 10.4471/redimat.2019.2822

It is imperative to expand upon earlier studies in mathematics education to confirm and extend knowledge in the field. Unfortunately, a paucity of such studies provides reason for the limited impact mathematics education has had on practice and policy (Lesh & Sriraman, 2010). According to Lesh and Sriraman, "[The] lack of accumulation is an important issue because most realistically complex problems will only be solved using coordinated sequences of studies, drawing on multiple practical and theoretical perspectives" (2010, p. 139). The aim of this study is to accumulate knowledge by re-examining various theories of students' development of multiplicative reasoning and, in particular, single-digit multiplication. A number of studies on students' strategies for single-digit multiplication have been conducted. However, researchers differ significantly in the strategies found and in the terminology they use (Sherin & Fuson, 2005). Although most studies on students learning single-digit multiplication are exemplary of the cognitive/rationalist approach based on experiments or interviews with individual students, little attention has been given to the broader cultural or instructional contexts of these studies (Verschaffel, Greer, & DeCorte, 2007). Although there is an increasing number of studies on the impact of instructional and cultural environments of this development of students' single-digit multiplication knowledge and understanding, there remains a lack of research on how students develop their understanding of multiplication in classroom settings by examining subsequent classroom interactions (Kaufmann, 2010). The sociocultural perspective proposes that, students' reasoning about multiplication cannot be separated from their participation in the interactive constitution of *taken-as-shared* mathematical meanings in a classroom. In a broader context and from a Nordic perspective, several important empirical studies on classroom interaction and the role of students' contributions in mathematical whole-class discourse have been conducted (Emanuelsson & Sahlström, 2008; Ryve, Larsson, & Nilsson, 2013; Streitlien, 2009). By analysing data from 24 classroom lessons on teaching and learning multiplication in eight third-grade classes, the aim of the study is to contribute to research in this field and conceptualisations of students' reasoning for multiplication.

The following research question guided the study:

How do students encounter and reason about multiplication in a primary school classroom context?

Literature Review

A plethora of studies on students' strategies for single-digit multiplication have been conducted (Anghileri, 1989; Greer, 1992; Sherin & Fuson, 2005; Vergnaud, 1988). However, the strategies and terminology employed differ significantly among researchers (Verschaffel et al., 2007). To classify research on students' learning of single-digit multiplication, the literature review includes three different threads of research: semantic types, intuitive models and solution procedures.

Research on semantic types comprises classifying different multiplicative situations described in word problems according to how they are schematised prior to the solution. After reviewing studies conducted in the 1980s and early 1990s, Greer (1992) proposed a synthesis of semantic types for multiplication, namely, *models of situations* and presented a framework for the analysis of multiplication of positive integers. He furthermore stated that the most important classes of situations involving the multiplication of integers include equal groups, multiplicative comparison, Cartesian products and rectangular areas (Greer, 1992). A valuable framework for research is provided by this structural analysis of semantic types. Mulligan (1992) followed 70 girls through grades two and three, from the stage they had no formal instruction in multiplication to when they were taught basic multiplication. Their solution strategies for a variety of multiplication word problems were analysed during four interview stages. The word problems were classified according to differences in semantic structure. The students answered the word problems containing repeated addition and rate most correctly, while experiencing the most difficulties with word problems containing a Cartesian product. Furthermore, studies on how the textbooks treat these kinds of sematic types were also conducted. Several potential factors influence students' reasoning of multiplication; textbooks are often an important source (Sidenvall, Lithner, & Jäder, 2015). In Norway, Kaufmann (2010) showed that textbooks emphasise equal grouping while other categories were only sporadically presented. Despite the distinction between models of situations, which is important pedagogically and provide an analytical framework, this article focuses on how students reason about multiplication in a classroom setting.

Research has identified intuitive models, which constitute and affect students' reasoning (Fischbein & et al., 1985; Mulligan & Mitchelmore, 1996). Discussion on intuitive models can be traced to Fischbein and colleagues (Sherin & Fuson, 2005). Fischbein et al. (1985, p. 4) hypothesised, "Each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model." The participants, 623 grade five, seven and nine Italian students, were asked to choose the operation needed to solve 12 multiplication word problems. These word problems contained factors with whole numbers, whole numbers and decimal numbers and decimal numbers. Fischbein et al. (1985) revealed the role of the decimal in the structure of a multiplication problem is clearly decisive in retrieving the correct operation, and concluded that a primitive model of multiplication is repeated addition. This result was in contrast to Mulligan and Mitchelmore's (1997) findings. They interviewed 60 girls when they were in grade two and had been taught basic addition and subtraction, but not multiplication, and in grade three when they had been taught basic multiplication. They solved the same set of word problems based on Greers' (1992) categories. They used three main intuitive models. First, direct counting was employed by using either cubes or visualisation and, subsequently, counting the cubes one by one. Second, repeated addition was used by rhythmic counting, skip counting, or additive calculation that takes advantage of the equal-sized groups present in the word problem. Third, they used multiplicative operation, a model that was inferred when students gave correct responses without appearing to form the entire sequence of multiples. By considering these results, the teacher's task is to assist students to widen their repertoire and reasoning about calculations strategies. For example, students who can solve a variety of multiplicative problems by direct counting may be encouraged to use equal group structure in their reasoning to develop more efficient strategies involving repeated addition.

Analyses of solution procedures describe the sequence of operations that a student performs from the given numbers to the product. Primary classroom mathematics often focuses on oral communication. For example, the teacher may ask students if they can explain their answer and accordingly, the solution procedures become central. Developing other researchers' work and their own data and analysis, Sherin and Fuson (2005) proposed taxonomy of strategies for single-digit multiplication. They interviewed third-grade students before, during and after multiplication instruction. Sherin and Fuson (2005) noted a *count-all* strategy. Students count from one to the product as they perform the computation. All values between one and the total are represented. This category of strategies is the largest and most varied part of their taxonomy. The varieties include students' use of drawing, *count-all-paper-based* and then use the figure to count the elements from one to the total in the figure. This strategy is closely related to the use of visual tools (Kaufmann, 2010). The presence of visual tools like triangles results in students counting the triangles one by one, representing all the values between one and 12 (Fig. 1).

Figure 1. Student points at the triangles and counts them one by one

Additive calculation is another strategy (Sherin & Fuson, 2005). Students' prior learning experiences in addition ensure these existing resources provide the basis for strategies that are less time-consuming and easier to enact than count-all strategies, which are based on addition-related techniques. A subtype of this category is *repeated addition.* The students perform sequential additions, each time adding the group size to the current value of the total. In Norwegian textbooks, this strategy is emphasised in the introduction of multiplication in the third grade (Kaufmann, 2010) (Fig. 2).

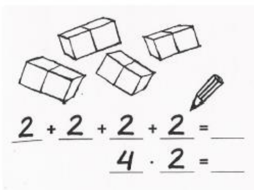

Figure 2. An example of the connection between repeated addition and multiplication

Another strategy described in Sherin and Fuson's (2005) taxonomy is *count-by*. When the multiplication instruction begins, students start the extended task of learning various number-specific computational resources that can support more efficient and accurate strategies. They learn sequences such as 6, 12, 18 …, which make the class of computational strategies labelled count-by strategies possible. An example includes multiplying 4 x 8 by counting in fours. According to Steel and Funnell (2001), who examined strategy use in a group of 8–9 year-old students in the United Kingdom, *count-by* was the slowest strategy characterised by the most student errors. However, Kaufmann (2010), who studied Norwegian third-grade students, found that students often used *count-by* to solve multiplication tasks and hardly made errors. This is possibly because teachers emphasised the *count-by* strategy and students learned songs in a *count-by* sequence. Often using these songs to solve multiplication tasks.

The strategy, *learned product* is associated with a large collection of number-specific resources: The multiplication triads (Sherin & Fuson, 2005). Learning these multiplication triads typically demands much student time and effort and, thus, they claim to know the answer. However, Kaufmann (2010) found several episodes in classroom discussions where students used the *learned product* strategy to solve multiplication tasks without a proper context and failed when they had to relate multiplication to a context. For example, many students using *learned product*, suggested nine when the teacher showed them three pencils in each of her hands (Fig. 3). Finally, one student said, "*If you add the pencils you get six, but if you multiply them you get nine*."

Figure 3. Two pairs of three pencils, illustrating two times three

Much of the classroom activity in primary school is performed orally through discussions about tasks and their solutions, either between the teacher and students, or among the students themselves. Previous studies have revealed various features are beneficial to explain students' reasoning about multiplication (Kaufmann, 2010). However, when studying aspects of students' multiplication reasoning in classroom settings, the data may reveal new aspects as shown in the results section.

This literature review has revealed a divergence on the types of strategies and terminology employed to describe these types. There is a lack of research on multiplication in a classroom setting (Verschaffel et al., 2007). The contribution made by this study is in describing and characterising students' reasoning on multiplication by studying classroom interactions. A minimal number of studies have focused on the development of student's strategies for multiplying single-digit numbers (Sherin & Fuson, 2005; Verschaffel et al., 2007). The results of this classroom study are related to other studies of students' strategy use in multiplication.

Theoretical Framework

The purpose of the study was to explore how students reason when working with multiplication in a classroom. A sociocultural perspective (Vygotsky, 1978) was employed to study students' reasoning in multiplication. Words and linguistic expression, among other semiotic tools, allow students to communicate knowledge and insight with each other. Thus, conversations are the most important learning arena (Säljö, 2005). When students learn to handle multiplication concepts, they learn how to reason and how conceptual content is determined within a particular practice, such as in schools. A sociocultural perspective views learning as the process of individuals appropriating knowledge and skills to which they are exposed. The direction taken by appropriation (Säljö, 2005; Wertsch, 1998) is neither linear nor easy; it includes a tension between the tool, developed and understood in a broader culture, and how it is utilised in a particular context. An important and fascinating aspect of appropriation is, when a person acquires and uses a new tool it always involves a conceptual tension or even resistance (Wertsch, 1998). A tool offers a meaning that has to be adapted to a specific case or a concrete problem. How the cultural tool is used is often not a decided and fixed matter, especially when users are not accustomed to it. Students often quickly alternate between various multiplication strategies; sometimes they connect these, while at other times they focus on the differences. In a classroom context, mastery of multiplication involves the process of change in which students adopt their language as it functions in multiplication and manage to utilise the physical tools in the classroom. In this appropriation process, students encounter a certain tension in the tools being used. Although students master the learned product in multiplication, they must also interpret the activities and use the tools in a multiplicative context. The conditions involving the use of tools in the teaching and learning process need to be understood clearly (Radford, 2012). This is not evident.

The significance of understanding students' reasoning processes which occur in an institutional setting is imperative. When students reason about multiplication and give seemingly absurd responses, this must be viewed as rational from a situated perspective. The meaning-making practices students engage in are not irrational. Students become acquainted with *doing school* through their own experiences. Through such socialisation, they learn how mathematics tasks and communication are normally organised (Lantz-Andersson, Linderoth, & Säljö, 2009).

Primary school students experience reasoning as an essential part of mathematical classroom activities. Through conversations with one another and/or the teacher, they explain their ideas, concepts and connections. When students are working on multiplication, this interaction is important, affecting the direction of the conversation. If some students focus on repeated addition as a solution method, more students will attempt this method in the next activity (Kaufmann, 2018). When the teacher asks students for suggestions on other solution strategies, these are usually about examples discussed among the class. Students build their knowledge based on that of other students. To understand how students' reason about multiplication, the interactions and conversations that take place in the classroom around multiplication should be studied.

Method

This study was based on the view that reasoning in the classroom is a process of students' appropriation of multiplication. The use of discussion as a tool to increase reasoning has gained emphasis in classrooms worldwide, including those in Norway (Ludvigsen et al., 2016). A central part of primary school mathematics teaching in Norwegian classrooms involves the teacher discussing tasks and their solutions with students (Streitlien, 2009). Teachers often ask students how they solved the task. Therefore, to examine how students reason about multiplication in school, it is necessary to follow students' discussions about multiplication. To collect beneficial data to examine reasoning on multiplication, three consecutive, introductory multiplication lessons were video- and audio-recorded in eight classrooms. The video camera was located at the back of the classroom in order to obtain a good view of the teacher and students. An audio recorder was attached to the teacher in the event that the video recorder was unclear. Seven teachers from five schools participated in the study. One teacher taught mathematics to two different groups in the third grade; both groups were involved in the study. The total video recording time in each class varied between 98-224 minutes, with an average of 156 minutes per class.

Data Analysis

From a sociocultural perspective, classroom practices are not regarded as a window to capture individual cognitive processes; rather, these practices are viewed as a participation process in classroom activities. Consequently, a suitable unit of analysis to capture the reasoning processes constituted the interactions that occurred when students were involved in multiplication. These interactions included student dialogues, activities and discussions about the tools they used in these activities. Therefore, the data analysis comprised the teacher and her students working on multiplication in class.

First, the data were collected. Subsequently, the data were transcribed. These transcriptions contained all verbalisations between the students in the classroom; most often between the teacher and students, but also specifically between students when they were working in groups; we followed one of the groups. Video camera footage was also attached to the transcriptions for clarification purposes. Subsequently, all the transcribed situations wherein the students explained, clarified or discussed different tasks, solutions or concepts of multiplication were marked. If students provided a one-word answer, the teacher asked them to explain how they got to the answer. Then the data were classified (Alvesson & Sköldberg,

2009) according to the types of strategies/solution methods that were discussed, such as repeated addition. Thereafter, the categories were considered with colleagues, and further developed and refined. The result was a taxonomy with seven different categories. These categories, the different classes and the lessons are provided in the appendix. Some of the categories concurred with those in the literature.

Results

The findings revealed that some of the students' multiplication reasoning in classrooms, in accordance with previous findings, included the count-all, count-by, repeated addition and learned product strategies (see appendix). Two reasoning procedures that the students used were different or not examined in previous research. *Students discuss different characteristics of multiplication* had not been examined previously, whereas, although *additive calculation* is included in Sherin and Fuson's (2005) categories, the results of this study found aspects thereof that had not been specifically highlighted.

Students' Reasoning About Different Characteristics of Multiplication

During the first lesson on multiplication in Dorte's class, she wanted to show the relationship between repeated addition and multiplication. She wrote 1 x 2 = 2; 2 x 2 = 2 + 2; 2 x 3 = 2 + 2 + 2 on the blackboard. Subsequently, she illustrated the next example, 2 x 4, with pencils. Accordingly, she held four pencils in each hand (Fig. 4). A classroom discussion on whether the right answer was 4 x 4 or 2 x 4 followed.

18.02	*Dorte*	Do you know? You said four and four. I watched you previously, since what you thought was right. But four and four. That is actually four first and four afterwards. Mmm. You had thought quite right (3) What is happening? I need your attention.	Dorte has written 4 x 2 on the blackboard.
18.26	David	Dorte. Dorte. Can I say something about times multipli-	
18.34	*Dorte*	Multiplication. Mmm.	

18.35	David	Eh, if you count with that number first. You should actually count only (3) how many times you should take the other number. And if you reverse the numbers in the arithmetic problem, it will be precisely the same.	Dorte writes 2 x 4 on the blackboard while David is talking.
18.51	*Dorte*	Exactly. Do you know? No, we put away one pencil from each hand.	All the students have pencils.
18.56	Stud*	There will only be one left.	
18.58	*Dorte*	No. You should have four in each hand; do you know? You should have four in each hand.	
19.03	David	Dorte. Dorte. Dorte. The first number proves only how many times we should take the other number. You should not calculate with the first number in a way that you should not add it to the rest. I figured that out when I made this multiplication table.	
19.27	*Dorte*	Table, yes.	
19.28	David	Table at home.	
19.29	*Dorte*	Yes (8). If Derek takes three pencils in one hand, and Doris takes three pencils in another hand, you can solve it.	

____

* When several students talk at the same time, it is difficult to find out who says what. These statements have, therefore, been attributed to “Stud” in the transcriptions. The numbers enclosed in parentheses indicate the length (in seconds) of the pauses.

Excerpt 1. Students discuss different characteristics of multiplication

Figure 4. Four pencils in each hand

There was a discussion on whether four pencils in each hand equalled 4 x 4. The students were also uncertain about the relationship between repeated addition and multiplication. Some suggested that 3 x 3 was the same as 3 + 3. Dorte sought an explanation of why the example with the pencils could be written as 4 x 2 (164–168). David interrupted and explained the significance of the digits in multiplication (172–178) and the roles of the multiplicand and the multiplier. Meanwhile, Dorte wrote 2 x 4 on the blackboard. David noted the answer would be the same if the numbers in the arithmetic problem were reversed (176–178), referring to the commutative law of multiplication. Dorte continued with the pencils (179-181, 183-185). David explained the difference between multiplication and addition (186-193) and the two important aspects of multiplication. He clarified the difference between multiplication and addition; when someone multiplies numbers, one cannot just add them. He further described the commutative law of multiplication. He not only recognised the communicative law of multiplication by reproducing that law, but explained concepts about the multiplicative law in this activity in his own words.

The situation in the second excerpt varies from the first one. In the first situation, Dina allowed her student to complete his reasoning about multiplication. However, in the second situation, the teacher interrupted her student. The following excerpt is from the beginning of the third lesson on multiplication in another class. The teacher had started her lesson by relating multiplication to what they had done in previous lessons and suddenly asked who experienced multiplication as difficult.

01.46	*Dina*	Do you know that you are able to multiply?	Dina turns to the whole class. Several students respond with yes at the same time.
01.48	Stud	Yes.	
01.50	*Dina*	Is there someone that finds this very difficult and does not understand what we are doing? This is okay because we should continue with this for a long time and for several years. It is very advisable if you let me know now if you do not understand what we are doing…Dan?	

02.10	Dan	That. Ten times ten times ten is a thousand.
02.16	*Dina*	Wow…Have you figured out this by yourself?
02.20	Dan	No.
02.21	*Dina*	No? How have you figured out that?
02.22	Dan	Eh. Because. Ten times ten is a hundred…and then. You could. Count. To…to a hundred ten times.
02.35	*Dina*	And then you get thousand. Very good. We have not learned large numbers yet.
02.40	Dan	No. We got-
02.42	*Dina*	That was very good. Della?
02.44	Della	I do not remember.
02.46	*Dina*	You do not remember what you should say. No. Okay. Then. Daniel, can you find three pencils for me?

Excerpt 2. Students discuss different characteristics of multiplication

The teacher's statement that multiplication can be difficult (034-042) resulted in Dan's response. It was evident that Dan had used multiplication in other situations. He used an example of numbers larger than those they had used at school. Furthermore, his example involved three factors (043-044), which had not been taught. He had not worked this out himself (045-047). He explained how he had arrived at the answer (050-053). The teacher confirmed the answer (054) and explained that they had not yet learned large numbers (055-056). She spoke to another student who had a different question (058). Dan attempted to respond (057) to the teacher's statement that they have not learned large numbers, but she interrupted him.

The analysis of the students' discussions with the teacher and other students about various characteristics of multiplication revealed general reasoning with regard to multiplication. They broadened the context by moving beyond the actual example by focusing their attention on mathematical relationships. The students made sense of the multiplication activities. Examples of appropriation included participating in activities, and using their own words, perspectives, goals and actions. This was not reported in the literature. According to Sherin and Fuson (2005), the ability to multiply using the learned product strategy is the most important aspect of different multiplication strategies: "[…] but broad learning of multiplication triads is likely to be dependent on substantial instructional focus. Thus, broadly speaking, classrooms and cultures that mobilise organised and sustained efforts for such learning will be more successful" (p. 383). However, the results of this study suggested the students wanted to describe what multiplication means through conversations/interactions with the other students. Their ability to talk about the activities implies that they regulated their own activities through concepts (Säljö, 2005) and structured their own thinking on whether a problem requires multiplication. Therefore, students no longer automatically responded to a situation, but classified it as multiplication.

Additive Calculation

The students' first lesson on multiplication in Anita's class is described below in Excerpt 3. They had worked without the textbook. Activities had focused on the grouping of equal amounts. In the following Excerpt (3), the students and teacher discussed a picture of a birthday party in the textbook's multiplication chapter: Five sodas with one straw in each were illustrated.

25.09	*Anita*	Yes (3) instead of one straw in each, we decide that there are two straws in each glass. How many straws will there be? (11). There are two straws in each glass; how many straws do you have? (19). Ada?	Originally, there was one straw in each bottle, but Anita asks what the answer will be with two straws in each bottle.
25.53	Ada	Ten.	
25.54	*Anita*	Yes. How did you solve that?	
25.56	Ada	I count two, four, six, eight. Eight, ten.	
26.00	*Anita*	Can you write that on the blackboard?	
26.10	Ada	What should I write?	
26.11	*Anita*	Hmm?	
26.12	Ada	What should I write? Should I write two, four?	Ada writes 2 + 2 + 2 + 2 + 2 =

26.15	*Anita*	Mmm, two and a plus between; can you write that? (38). Thank you (8). You, are there other ways of thinking? It is not finding the most difficult way. Aksel?	10 on the blackboard.
27.11	Aksel	Five plus five.	Aksel writes 5 + 5 on the blackboard.
27.12	*Anita*	Yes. [Oh, that was what I was going to say.]	
27.16	Stud*	But I know a different one. [Me, too.]	
27.19	*Anita*	Do you? (17). Thanks. Atle?	
27.38	Atle	One plus one plus one till I arrive at ten.	
27.42	*Anita*	Mmm. Ari?	
27.43	Ari	One plus one plus one plus one plus two plus two plus one.	
27.46	*Anita*	Arnt?	
27.48	Arnt	Four plus six.	
27.50	*Anita*	Mmm. Yes.	
27.54	Stud	Ehh ohh ehh, three plus two, two plus three.	
27.59	*Anita*	Heheh.	
28.00	Stud	And so on.	
28.01	*Anita*	Yes, but now you start to make (3) it more difficult than it is, isn't it? [Two plus three.] Yes, now you are doing such difficult tasks.	
28.10	Arild	I know a simple one.	
28.11	*Anita*	Arild?	
28.12	Arild	Eight plus two.	
28.13	*Anita*	Yes, but now-	
28.15	Stud	And nine plus one.	In this sequence, a lot of students are speaking in chorus.
28.16	*Anita*	But that does not -	
	Stud	[Yes, that was easy.]	
28.18	*Anita*	Eight plus two is like Ada's [answer.]	
	Stud	[Ten plus zero.]	
28.20	*Anita*	Then you have first added two and two becomes four, plus two is six, plus two is eight, plus two is ten … But listen. Altogether, all the children. They have clothes.	Anita refers to another context in the picture.

*When several students spoke at once, it was difficult to determine who said what. These statements have been attributed to "Stud" in the transcriptions.

Excerpt 3. Additive calculation

Anita moved on from the example in the textbook and asked how many straws there would be if each soda had two straws. (227–229). Ada said ten (232) by employing number counting (234-235). Her answer corresponded to the actual problem. Anita asked for more suggestions (244-246). Aksel's suggestion of 5 + 5 = 10 did not concur with (247) the actual problem. Anita did not comment, but asked Atle to explain (253). His suggestion did not correspond with the example. Atle also used addition with an equal group size; similar to the previous suggestions. Ari (257-258), Arnt (260) and another student (262-263) each suggested addition with different group sizes. It became a game with numbers that provided the sum of 10 with various addends. Because the seemingly absurd responses were related to multiplication, the meaning-making practice the students engaged in was not irrational. The students were used to *doing school* through their own experiences. Through this socialisation they appropriated the discussion about different solutions. Anita expected the students' answers to be related to multiplication and repeated addition. However, some of the responses with repeated addition did not match the example. Anita resolved the tension by stating they were suggesting difficult calculations (269-270). Arild (273) and other students (275-280) made suggestions that involved easy calculations. Anita did not comment on the student's last reply (280), but responded to Arild's answer of "eight plus two" and attempted to link this (281-283) to Ada's earlier response. Without concluding the problem, Anita proceeded to the next task (284-285).

Sherin and Fuson (2005) applied this strategy in their *additive calculation*. However, they revealed a different meaning in this category than that shown in this study: Students solved 4 x 3 by adding 3 + 3 = 6 and, subsequently, adding 3 to their answers. This is possibly because they had mastered addition. Using *additive calculation* was meaningful for the students. Furthermore, this reasoning procedure is common when students reason about multiplication. However, as revealed in this study, students may use different sub-totals, which is common in addition. This means that the students deviate from using equal numbers. This solution strategy has not been revealed in previous studies. It may depend on the lack of focus given to how students reason about multiplication in a classroom setting.

The meaning-making practices that students engage in are not irrational. Instead, students' reasoning is a response to what Brousseau (1997) called the didactical contract; the rules of communication established in educational settings, that participants learn to identify and use as structuring resources. Teacher and student dialogues do not automatically lead to mathematically founded reasoning or deeper learning (Emanuelsson & Sahlström, 2008). The teacher and students experience the meaning created in the relationship between addition and the glasses with straws differently. There is tension in the dialogue because the teacher's interpretation of the situation differs from those of the students. In this example, Anita focused on different combinations using repeated addition with ten as the answer. The students played a game with numbers; the answer of each was ten. Anita remarked somewhat imprecisely that the students made the solution more difficult than it was. The students' response was to perform easy tasks with two addends, without considering the equal group size. Anita did not make definite comments on all these solutions, but attempted to link Arild's suggestion to Ada's response by saying, "Eight plus two is like Ada's [answer]". It may also be discussed whether 2 + 2 + 2 + 2 + 2 and 8 + 2 are the same.

Summing Up

The analysis of the students' discussions with the teacher and other students about various characteristics of multiplication revealed general multiplication reasoning. They broadened the context by moving beyond the actual examples provided. They wanted to describe what multiplication involved through conversations/interactions with the other students. They possibly attempted to explain similarities and differences between repeated addition and multiplication. Some students may have discovered how the commutative law applies to multiplication and wanted to explain this to the teacher and the class. They expanded multiplication and related it to previous tasks. This process can be described as appropriation, which involves "taking what someone else produces during [a] joint activity for one's own use in [a] subsequent productive activity while using new meanings for words, new perspectives and new goals and actions"

(Moschkovich, 2004, p. 51). Appropriation involves more than copying a tool or simply mechanically applying it. It is a subtle process in which students need to learn the relations among the different tools and use these tools in concrete situations and problems. This analysis also revealed examples where the students used addition with unequal group sizes. The teacher wanted to focus on equal groups (repeated addition), while the students suggested the addition of varying group sizes that resulted in the same answer. While they should have justified their answers, instead they suggested the addition of different group sizes. This strategy was not noted in the literature. This category is important to understand what happens in a classroom. However, it may prevent students from understanding multiplication because it is not related to the concept. A common perspective is that because of students' prior learning experiences in addition, these existing resources can provide the basis for strategies that are less time-consuming and easier to enact than count-all strategies. These strategies are based on addition-related techniques. It may be due to the students' previous experiences with addition, because they have mastered this strategy. Using addition to arrive at the answer is meaningful to them. However, several students begin to use different addends; a technique related to addition. However, students deviate from using equal numbers.

Discussion and Conclusion

The aim of this study was to accumulate knowledge by re-examining theories of students' development of multiplicative reasoning, in particular, single-digit multiplication. This goal was achieved by studying classroom interactions. This study's contribution is its analysis of the interactions among students in a classroom during their learning multiplication in the third grade. Specifically, students' reasoning procedures were examined. The results demonstrated reasoning procedures students employed in multiplication that have not been examined previously.

Students' reasoning about multiplication appears to be strongly influenced by two aspects of instructional practice: Discussions about multiplicative tasks/problems and how these problems are conceived and treated by the teacher. The general research question is how students encounter and reason about multiplication in primary school? While students choose to move on from discussing examples, general multiplication reasoning about multiplication has not been reported in the literature. The reason this category is evident in this study may be methodological or theoretical. Methodologically, this study focused on students' reasoning about multiplication in a natural classroom setting, while many studies have employed experiments and/or interviews with individual students. Theoretically, a description of the involvement process is when students try to master the mediating tools used in the classroom. According to Wertsch (1998), appropriation can be perceived as the process of how people acquire understanding and knowledge in sociocultural practice. Consequently, formulating utterances involves the process where students appropriate words from others and make them, at least in part, their own. Consequently, students must articulate what multiplication means and identify situations that can be described with multiplication. The central aspect of the analysis was the difference between solving a problem when the teacher asked the students a question and being able to talk about multiplication. The emphasis in this study was on the importance of the *students discussing different characteristics of the multiplication* procedure and the lack of this perspective compared to other research reports about multiplication. A characteristic of this reasoning procedure is that the students can reason about multiplication in a context larger than just providing an answer to a specific multiplication task. One aspect when students are working with multiplication in the classroom is that they can perform multiplication in activities where we can observe this. Another aspect is when students are reasoning about multiplication and actively structure their thinking. This implies reasoning about different characteristics of multiplication; not just performing multiplication tasks but having discussions. This reasoning process is important because reasoning and discussing activities implies that you regulate your own activities using concepts (Säljö, 2005). Consequently, students no longer automatically respond to situations, but classify them as multiplication.

The sociocultural perspective proposes students' reasoning about multiplication cannot be separated from their participation in an interactive constitution of *taken–as-shared* mathematical meanings. The teacher plays an extremely important role in the development of such a micro-culture (Verschaffel et al., 2007). One role of a teacher is to orchestrate a mathematical classroom as a place where students carry out, discuss and justify solution procedures for mathematical situations (Streitlien, 2009), in which their ideas and contributions serve as the departure point of a whole-class discourse. Balancing content and students' participation in a mathematics classroom is complicated. According to Ryve, Larsson and Nilsson (2013, p. 102), "There are still many complicated relations between students' engagement in the classroom, the teacher's way of orchestrating whole-class interaction, and how content is made explicit in the interaction." The excerpts revealed several examples of balancing content and participation. This was particularly evident in Excerpt 3, where the students' participation was at the expense of content. The teacher accepted the solutions as she allowed the students to participate in the discussion. The students' participation largely determines the co-construction of mathematical content and, in this case, resulted in a "down-grading of the mathematical complexity" (Emanuelsson & Sahlström, 2008, p. 212).

The absence of cumulative, well-developed knowledge about the practice of teaching and the limited links between research and practice has been major impediments to creating a working system of school mathematics (Ball, 2003). This study has attempted to meet this requirement. Furthermore, this study can be referred to as an empirical accumulation of students' strategies and reasoning procedures in multiplication. Some of the study results concur with previous findings. The other results have expanded knowledge of students' mastery of multiplication. Accordingly, this study could be considered a cumulative contribution to theories and frameworks for characterising students' encounter with, engagement in and appropriation of multiplicative reasoning.

Didactical Implications

There appears to be a pedagogical dilemma in the literature concerning students' understanding of models for multiplication and recommendations for teaching. There is consensus on the equal groups model as one of the most accessible models for young students. However, the introduction and extensive use of the equal groups model is reported to reduce multiplication to repeated addition, which does not support multiplicative reasoning. It is crucial that students consider the multiplicand (a specified number or quantity) and multiplier (a specified number of times). The abstract nature of the multiplicand and the multiplier is central, as they refer to different types of objects. It was evident from the study that students do not have sufficient experience with the difference between the multiplicand and multiplier.

Students should gain experience with multiplication as an effective operation at an early stage in their education. There may be a danger that students will find it easier to use addition rather than multiplication in simple tasks and activities they do at school. There is, thus, a possibility that they are not able to master multiplication, since they do not see the usefulness of it. In the classes observed, there were several cases where the students did not accept the teacher’s explanation that multiplication is an easier way to add. This was especially true when the numbers were small. Whether multiplication is easier than addition is based on the feelings and experiences of the individual. Consequently, it may become strange when the introduction to multiplication is about “multiplication is an easier way to figure out when we are going to put together the same number many times.” Although addition can be used in simple-number assignments, it is important to focus on how multiplication can appear to be a useful operation. Repeated addition should be used only as a means to generate the response to a multiplicative operation. Therefore, students should work with larger numbers so that they experience the benefit of using multiplication as an operation. Furthermore, one should use realistic tasks from daily life, although these may be difficult and involve larger numbers.

It must be understood that in an educational context there is a big step between solving a single problem in multiplication under the teacher's guidance and learning to reason about multiplication. There is a distinction between solving a problem and learning to multiply. It is common for the teacher to present the students with a situation in the class that they can discuss. This situation should be such that it provides experiences that can be described mathematically. The teacher wants the students to learn how to multiply in situations that can be described by multiplication. However, there is a difference between responding to problems in multiplication and learning to multiply by using it in other problems and activities. As noted from the analysis and results, there were mostly discussions about the answers to the different tasks and how the students intended to reach the answer. It is more important to be able to reason about the properties of multiplication and to use multiplication. In other words, the students must explain what multiplication means and what situations can be described by multiplication on their own. Teachers can facilitate such teaching, where students are challenged in this area, and where the teaching leads to investigative activities.

References

Alvesson, M., & Sköldberg, K. (2009). *Reflexive methodology: New vistas for qualitative research* (2nd ed.). London: Sage.

Anghileri, J. (1989). An investigation of young children's understanding of multiplication. *Educational Studies in Mathematics, 20*(4), 367–385. doi: 10.1007/BF00315607.

Ball, D. L. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education*. Santa Monica, CA: RAND.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Hingham: Kluwer Academic Publishers.

Emanuelsson, J., & Sahlström, F. (2008). The price of participation; teacher control versus student participation in classroom interaction. *Scandinavian Journal of Educational Research, 52*, 205–223. doi: 10.1080/00313830801915853.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*(1), 3–17.

Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 276–295). Charlotte: Charlotte, NC, USA: Information Age Publishing.

Kaufmann, O. T. (2010). *Students first meeting with multiplication in primary school: A sociocultural aprroach to appropriation of multiplication in the classroom*. University of Agder. Kristiansand.

Kaufmann, O. T. (2018). The problem of distinguishing multiplicative from additive reasoning in primary school classroom context. *European Journal of Science and Mathematics Education*, *6*(3), 100–112.

Lantz-Andersson, A., Linderoth, J., & Säljö, R. (2009). What's the problem? Meaning making and learning to do mathematical word problems in the context of digital tools. *An International Journal of the Learning Sciences, 37*(4), 325–343. doi: 10.1007/s11251-008-9050-0.

Lesh, R., & Sriraman, B. (2010). Re-conceptualizing Mathematics Education as a Design Science. In B. Sriraman & L. D. English (Eds.), *Advances in mathematics education* (pp. 123–146). Berlin: Springer.

Ludvigsen, S., Nortvedt, G. A., Pettersen, A., Pettersson, A., Sollerman, S., Ólafsson, R. F., ... Braeken, J. (2016). *Northern Lights on PISA and TALIS*. Copenhagen: Nordisk Ministerråd.

Moschkovich, J. N. (2004). Appropriating mathematical practices: a case study of learning to use and explore functions through interaction with a tutor. *Educational Studies in Mathematics, 55*(1–3), 49–80. doi: 10.1023/B:EDUC.0000017691.13428.b9.

Mulligan, J. (1992). Children's solutions to multiplication and division word problems: a longitudinal study. *Mathematics Education Research Journal, 4*(1), 24–41. doi: 10.1007/BF03217230.

Mulligan, J., & Mitchelmore, M. (1996). Children's representations of multiplication and division word problems. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning* (pp. 163–184). Adelaide: The Australian Association of Mathematics Teachers.

Mulligan, J., & Mitchelmore, M. (1997). Young Children's Intuitive Models of Multiplication and Division. *Journal for Research in Mathematics Education, 28*(3), 309–330. doi: 10.2307/749783.

Radford, L. (2012). On the Cognitive, Epistemic, and Ontological Roles of Artifacts. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to 'lived' resources: Mathematics curriculum materials and teacher development* (Vol. 7, pp. 283–288). Dordrecht: Springer.

Ryve, A., Larsson, M., & Nilsson, P. (2013). Analyzing content and participation in classroom discourse: Dimensions of variation, mediating tools, and conceptual accountability. *Scandinavian Journal of Educational Research, 57*(1), 101–114. doi:10.1080/00313831.2011.628689

Sherin, B., & Fuson, K. (2005). Multiplication Strategies and the Appropriation of Computational Resources. *Journal for Research in Mathematics Education, 36*(4), 347–395. doi: 10.2307/30035044.

Sidenvall, J., Lithner, J., & Jäder, J. (2015). Students' reasoning in mathematics textbook task-solving. *International Journal of Mathematical Education in Science and Technology, 46*(4), 533–552. doi:10.1080/0020739X.2014.992986

Steel, S., & Funnell, E. (2001). Learning Multiplication Facts: A Study of Children Taught by Discovery Methods in England. *Journal of Experimental Child Psychology*, *79*(1), 37–55. doi: 10.1006/jecp.2000.2579.

Streitlien, Å. (2009). *Hvem får ordet og hvem har svaret?: Om elevmedvirkning i matematikkundervisningen* [Who gets the word and who has the answer? About student participation in mathematics teaching]. Oslo: Universitetsforl.

Säljö, R. (2005). *Lärande och kulturella redskap: om lärprocesser och det kollektiva minnet* [Learning and cultural tools. On processes of learning and the collective memory]. Stockholm: Norstedts akademiska förlag.

Vergnaud, G. (1988). Multiplicative structures. In J. B. Hiebert, M (Ed.), *Number Concepts and Operations in the Middle Grades (Research Agenda for Mathematics Education)* (pp. 141 –161). Hillsdale, NJ: Lawrence Erlbaum.

Verschaffel, L., Greer, B., & DeCorte, E. (2007). Second handbook of research on mathematics teaching and learning. In F. K. Lester (Ed.), *Whole number concepts and operations* (Vol. 1, pp. 557–628). Charlotte, N.C: Information Age.

Vygotsky, L. S. (1978). *Mind in society: the development of higher psychological processes*. Cambridge, Mass: Harvard University Press.

Wertsch, J. V. (1998). *Mind as action*. New York: Oxford University Press.

Odd Tore Kaufmann is an assistant professor of mathematics education at Østfold University College, Norway.

Contact address: Direct correspondence concerning this article, should be addressed to the author. **Postal address**: B R A Veien 4, 1783 Halden, Norway. **Email:** odd.t.kaufmann@hiof.no

Appendix

	Count all	Addition	Repeated addition	Count-by	Collapse groups and add	Learned product	Students discuss different characteristics of multiplication
1.Lesson	A1 (2) C	A1 (3) A2 (7) B C (2) EV EL	A1 (2) A2 (6) B C (7) DI (9) EV (3) EL (4)	A1(3) A2 (6) B C (8) DO Di (2) EV (3) EL (7)	A1	A1 (2) B C (13) DO (5) DI (4) EV (8) El (2)	DO EV EL
2. Lesson	C	EL (4)	A2 (2) B C (4) DO (2) Di (2) EV (8) EL (14)	A2 (2) B C DI EV (2)	A2 B	A2 (2) B (5) C (7) DO (12) DI (5) EL (13) EV (11)	B (2) DO (2)
3. Lesson	Do EV	B (2) EV (15) EL (3)	A2 B C (2) DO (5) DI (2) EV (4) EL (16)	A1 B (3) C DO (2) DI EV (11) EL	B DO EV	A1 B (4) C (7) DO (12) DI (4) EV (12) EL (24)	C DO