Making Manipulatives for Mathematics Education

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ABSTRACT

This paper describes the efforts of an interdisciplinary team of researchers as they collaborated to create a digital fabrication curriculum module for mathematics teacher education. The initial four-day workshop design was piloted with five pre-service teachers. The design objective was to introduce digital fabrication techniques joint with mathematical concepts and the design of classroom activities to develop pre-service teachers' technological, pedagogical, and content knowledge. The workshops included activities to find, adapt, create, and share mathematical manipulatives using digital fabrication tools, techniques, and platforms. Manipulatives are tangible objects reifying mathematical concepts and one type of representation used in mathematics teaching. The paper reports on the design process and our design motivations to address contextual constraints and varying levels of exposure to digital fabrication for both pre-service teachers and teacher educators. The developed "find-adapt-create-share" framework for introducing digital fabrication was evaluated through researchers' self-reflection and pre-service teachers' feedback during concluding interviews.

CCS CONCEPTS

• **Applied computing** → Education.

KEYWORDS

digital fabrication, manipulatives, mathematics teacher education

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1 INTRODUCTION

We are undergoing an ongoing digitalization of teaching and learning practices. Students' access to and use of digital tools impact and fundamentally transform our educational practices [15]. As emphasized by policymakers, researchers, school leaders, and teachers alike, there is a need for teacher educators to adapt and integrate digital competence in the curriculum [3, 7, 19]. We answered the request for teachers' digital competence in our mathematics education program in several ways, most explicitly by introducing a novel master course on digital tools and teaching aids for mathematics teaching. Parts of the course are filled with acquiring skills in digital teaching tools and programming to support mathematics teaching. However, we are also interested in exploring how digital fabrication technologies could be integrated into the course. Digital fabrication (DF) is "the process of translating a digital design developed on a computer into a physical object" [2]. DF technologies such as 3D printers, laser cutters, or vinyl cutters have become increasingly popular and can be found at Makerspaces and FabLabs worldwide [30]. DF and making have been integrated into education centered around science, technology, engineering, arts, and mathematics projects. Papavlasopoulou [20] reviews current research in making and found that learning through making in art, design, and technology practice can provide fertile ground for developing STEM in education. The DF research community has made considerable efforts to support teachers in acquiring DF skills, focusing on how teachers can integrate DF and design thinking in a STEM classroom [9, 17]. There has been little focus on how teachers can utilize digital fabrication to produce materials that aid learning, such as manipulatives in mathematics education [26]. Manipulatives are tangible objects used in mathematical teaching, reifying mathematical concepts to support learning. This paper reports on our efforts to design a DF curriculum module for mathematics teachers' education, focusing on making manipulatives for teaching mathematics. We review previous research on DF for mathematics teachers and theories on teacher professional knowledge to inform how technology and digital competencies should be integrated into mathematics teacher education. Synthesizing our findings, we propose a "find-adapt-create-share" framework describing how DF can be introduced in mathematics teacher education for making artifacts that aid learning. We piloted a series of four workshops with five students enrolled in our teacher education program, in the following called pre-service teachers (PST), to assess our framework and inform the design of a DF curriculum module as part of a master course on digital tools and teaching aids for mathematics teaching. We conclude our paper with lessons learned from the workshops and advocate using the "find-adapt-create-share" framework as one approach to scaffold DF skills for mathematics teacher education.

2 DIGITAL FABRICATION FOR MATHEMATICS TEACHERS

Stigberg's review [26] of previous research on DF for creating manipulatives in mathematics education found 17 research articles describing manipulatives reifying mathematical concepts in geometry, algebra, and fractions, using predominately 3D printing,

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all published after 2015. Ford and Minshall [6] reviewed how 3D printers are used in education in general. They identified six categories: to teach students about 3D printing; to teach educators about 3D printing; as a support technology during teaching; to produce artifacts that aid learning; to create assistive technologies; to support outreach activities. Previous research reporting from professional development projects for mathematics teachers often has a STEM context and focuses on how teachers can integrate DF and design thinking in the classroom [26]. Instead, we are interested in how to introduce DF to PSTs to support their professional development and aid them in creating manipulatives for mathematics teaching. Ford and Minshall [6] found previous research aiming to produce artifacts that aid learning, e.g., anatomy and chemistry education. These produced artifacts bring new elements into view, concealed for digital representations. They are easy to reproduce and cost-efficiency. [8, 10, 21] present 3D models reifying complex mathematical concepts. Unfortunately, none of those reports from interventions for teacher education or evaluations from classroom use. We built on the existing body of knowledge highlighting the benefits of DF for making manipulatives reifying mathematical concepts and explored how PSTs can learn DF to produce such artifacts to aid learning and support their professional development as requested by Ford and Minshall [6].

Lassiter et al. [12] identified six critical categories of knowledge that an educator needs to successfully integrate digital fabrication into a learners' formal educational experiences: digital design and fabrication techniques, engineering fundamentals, application of the design process, project design and management, strategies to align student learning to benchmarks and to leverage standards for assessment, partnership and asset building and alignment, the larger context of digital fabrication in the making, tinkering, and fabbing communities as well as the interests of industry and national economy. However, they do not specify how these categories can be implemented in a DF curriculum. Hjorth et al. [9] describe a framework for educating the educator based on a design studio approach, including three types of activities: workshops and lectures, peer collaboration, and in-school practice. A mixture of literature on digital fabrication in education and pragmatist design literature was taught through lectures, group exercises, and pre-work in a series of workshops. However, the authors do not specify how these topics were introduced in the workshops in more detail. Ulbrich et al. [28] report on DF workshops for teachers. They divide workshops into two parts: first they present examples and demonstrations to inspire and motivate teachers; then they focus on providing teachers with hands-on experience in 3D modeling and 3D printing as well as finding and downloading free online models.

In summary, we found various examples of manipulatives for mathematical concepts produced using DF [8, 10, 21]. Nevertheless, there is sparse research into how DF can be introduced to teachers. We see three main concepts: learn DF tools and techniques through hands-on experience [28], design thinking through long-term projects, and integrate into a teaching context through in-school practice [9]. In our research project, we will focus on designing a DF curriculum module for mathematics teacher education, including students' learning objectives, overall module design, and design of individual workshops. We take a holistic stance on learning DF skills in the context of making manipulatives for mathematics education, merging DF tools and techniques with pedagogical thinking and mathematical concepts discussed in more detail in the next section.

3 TEACHERS' PROFESSIONAL KNOWLEDGE

Teacher education programs are often professional-oriented, integrating subject matter knowledge (e.g., mathematics) and pedagogical content knowledge (e.g., teaching and learning mathematics) as described by Ball et al. [14]. Similarly, Schulman [25] defines necessary teacher "know-how" as subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. Mishra and Koehler [11] add technology to teachers' professional competencies. They present a theoretical model called TPACK to describe teachers' technological, pedagogical, and content knowledge illustrated in Figure 1 The model presents the relationship between technological knowledge (TK), pedagogical knowledge (PK), and content knowledge (CK) needed for teaching. PK refers to different teaching methods or how to lead the processes of working. CK comprehends "knowledge of concepts, theories, ideas, organizational frameworks, knowledge of evidence and proof, as well as established practices and approaches toward developing such knowledge. Knowledge and the nature of inquiry differ greatly between fields, and teachers should understand the deeper knowledge fundamentals of the disciplines in which they teach" [11]. TK is defined as knowledge about new technology in education. It has been extensively used in research on technology education and teachers' professional development in a technological context [5, 22, 29]. A critical insight from the model is that technological knowledge alone is not enough in a teaching context. Teachers must be able to integrate technology with subject content and pedagogical knowledge to enhance students learning. All three dimensions are interrelated and overlapping in Mishra and Koehler's model [11], highlighting their relation. New technology can affect both content and pedagogical knowledge, opening up for new teaching opportunities. In our research project, we define DF skills for making manipulatives as TK, knowledge about mathematical concepts reified by manipulatives as CK, and knowledge of how to use manipulatives in the classroom as PK.

Digital competencies are defined as a key competence in education across Europe [3]. In the Norwegian context, digital competencies have become part of teacher education guided by a professional digital competence (PDC) framework. Nagel [18] describes that Norwegian teacher educators are expected to focus on the pedagogical use of digital tools and need an understanding of digitalisations' implications for epistemic practices. Brevik et al. [4] define these competencies as a transformative digital agency, the "competence in taking initiatives and transforming their practices by selecting and using relevant digital tools." In line with TPACK, PDC highlights the importance of the thoughtful integration of digital tools into teaching practices and the understanding of digitalization in education and society. Skills such as critical thinking and problem-solving, creativity and innovation, as well as communication and collaboration, are tightly coupled to digital competencies and PDC. Previous research on DF for STEM emphasizes these competencies [9, 12, 31]. Therefore, it is essential that even we include these skills

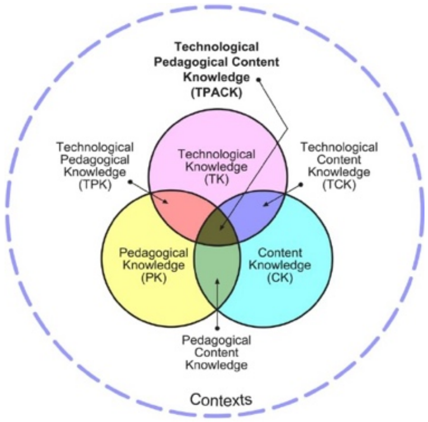

Figure 1: TPACK framework and its knowledge components by Mishra and Koehler [11]

as learning objectives in the curriculum and make sure to include them in the module's design.

How PSTs can acquire knowledge about DF in a mathematical pedagogical context can be described using Vygotsky's notation of the zone of proximal development (ZPD). According to Vygotsky, with help from a more knowledgeable person(s), people can appropriate knowledge they could not have learned by themselves [23]. To invite PSTs to work within the ZPD, Askew et al. [1] describe an approach by Bruner that reduces the impact of errors and scaffolds the students' work by reducing the amount of freedom in the beginning gradually increasing the complexity. This approach has been applied to teaching computational thinking and programming by Lee et al. [13] and Sentance and Wait [24]. Lee et al. propose a three-step approach called Use-Modify-Create, where learners start by using a ready-made program. In the second step, they read and modify the code, and in the third step, they create a new program, using acquired knowledge from steps one and two. Similar, Sentence and Wait propose a framework called PRIMM. They specify using code in more detail. First, learners should predict what a given code will do on execution, and then they run the code to test the prediction and investigate what happened. Both frameworks continue with a modify phase, and a create/make phase. We are inspired by these frameworks and will follow a scaffolding approach for introducing DF in mathematics teacher education.

4 TOWARDS A DF CURRICULUM MODULE FOR MATHEMATICS TEACHER EDUCATION

We investigate how DF could be part of a master course on digital tools and teaching aids for mathematics teaching. Derived from previously presented research on DF for mathematics education and the demand for the inclusion of 21st-century skills in teacher education, we define the following student learning objectives for the DF curriculum module in the master course:

- Students can produce manipulatives for mathematics teaching using DF technologies.
- Students have critical thinking and problem-solving skills for designing manipulatives and classroom activities.
- Students can collaborate in making manipulatives and are able to communicate their ideas and rationales to others

Furthermore, we design the DF curriculum module for mathematics teacher education with two core objectives:

- The DF module should introduce digital fabrication technologies integrated into the context of creating manipulatives for mathematics teaching, including both mathematical concepts and pedagogical thinking.
- The module should afford scaffolding, teaching DF with a low threshold and high ceiling. PSTs should produce a manipulative for a specific teaching activity from the start.

Based on the defined learning objectives and our two core objectives, we designed four scenarios of how DF could be used to produce manipulatives for teaching mathematics. These scenarios describe the need for different DF skills situated in potential teaching situations:

- A teacher is planning a mathematics class for the next school year. She wants to include more manipulatives in class, but there are limited manipulatives at school. She has no budget for buying commercial manipulatives, but the school has a newly created makerspace she can use. So, she thinks: "Maybe there is something on Thingiverse that I could download and make."
- A teacher found a manipulative for teaching angles online. She downloads the model and produces it at the school's makerspace. The manipulative includes 45, 90, and 180 degrees. She would like to add 60 degrees. She thinks: "Maybe I can use Tinkercad to add a new part to the manipulative."
- A teacher has a great idea for a manipulative demonstrating that the volume of a pyramid is a third of the volume of a cube. She cannot find something similar anywhere. So, she thinks: "Maybe I can use Tinkercad to create my own manipulative."
- A teacher creates her own manipulative for teaching fractions, and it works well in the classroom. She would like to invite her colleagues to use it as well. She thinks: "I have to show this to my colleagues at the next meeting. If they like it, I could even upload it to Thingiverse for others to use."

Using elicited DF skills described in the scenarios, we developed a four-component framework describing how DF can be introduced to PSTs to make artifacts that aid learning. As illustrated in Figure 2, we imagine three levels when making manipulatives for teaching mathematics arranged by increasing complexity from finding to creating manipulatives. Embedded in the described scenarios is the premise that technology use often involves finding available resources online. Finding and producing manipulatives from online repositories for an envisioned teaching activity is a first step in developing DF skills. Ulbrich et al. [28] list finding and downloading 3D models as one important task when introducing teachers to 3D printing. PSTs learn how to use online resources and reflect on how to adapt them to classroom activities with various levels of sophistication, from choosing a filament color or printing size to changing the digital model. PSTs need to acquire skills to modify or copy available manipulatives using different types of DF software at the adapt level. Finally, PSTs can apply acquired DF skills to create

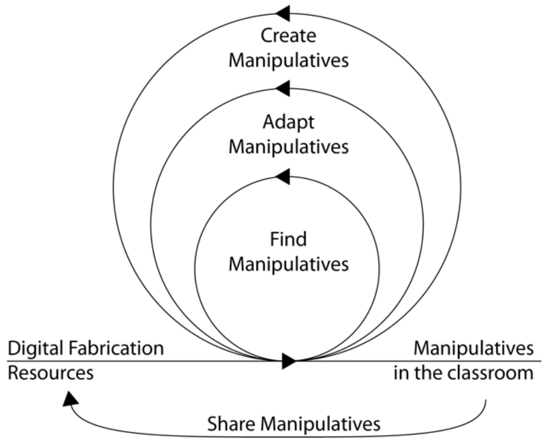

Figure 2: Find-Adapt-Create-Share Framework for Learning DF skills

new manipulatives for mathematical concepts inspired by available designs. At this create level, PSTs use DF tools and techniques to ideate new manipulatives in their design processes. Creating manipulatives almost always builds on a PSTs' previous experiences finding and adapting manipulatives. Each level is essentially a superset of what has come before. Finally, PSTs are encouraged to participate in maker culture and share their work with others to increase the number of available DF resources and thrive in maker culture, illustrated in Figure 2 by an arrow pointing from the created manipulatives for the classroom back to digital fabrication resources.

While we advocate using a "find-adapt-create-share" framework to describe how PSTs can develop DF skills and competencies, it is not intended to depict three distinct steps. In practice, we see no clean breakpoints among finding, adapting, and creating but gradual metamorphoses from one level to another. When designing a DF curriculum module, we need to pay attention to PSTs' previous knowledge and experiences to select the appropriate entry-level. The fourth component in the framework, sharing should be part of each level to aid learning DF skills and afford participation in a DF community. In the following, we describe how we have piloted the DF module as four workshops for PSTs to develop DF skills following the suggested framework.

5 FOUR WORKSHOPS FOR PRE-SERVICE TEACHERS TO DEVELOP DIGITAL FABRICATION SKILLS

We piloted a series of four workshops with five PSTs (four male, one female) in their fourth year of studies. The PSTs have a sound mathematical background and experiences from in-school practice in 5^{th}-10^{th} grade class. They completed a course in basic ICT skills and joined a field trip to a locale FabLab. They were recruited in a mathematics course, where we presented the project, and interested PSTs could sign up voluntarily. As compensation, they got lunch during the workshops, two cinema tickets, and a certificate of participation. The workshops were designed and conducted by an interdisciplinary team of researchers consisting of one associate professor in computer science with expertise in digital fabrication and design thinking; one professor and one associate professor in mathematic didactics with expertise in subject didactics, teacher education development, and professional development research; and one Ph.D. fellow with extensive experience as a mathematics teacher in primary and secondary education. We designed and conducted a total of four workshops, four hours each, once a week. We video-recorded the workshops to help us analyse, reflect on, and plan the workshops. In addition, the research group had weekly meetings between workshops to share reflections and discuss workshop implementations and results. Meeting minutes were stored in an online collaboration platform.

To elicit PSTs' feedback from the workshops, we conducted concluding individual interviews with all five PSTs asking about learning objectives, the design of the curriculum module, including the overall design of the workshops using the find-adapt-create-share framework, as well as the aim to introduce DF skills joint with mathematical concepts and the design of classroom activities to develop PSTs' technological, pedagogical, and content knowledge. The interviews lasted between 53-67 minutes. All interviews were fully transcribed and coded using a two-step inductive approach. First, we coded the interviews focusing on workshop design and learning objectives using Nvivo. Second, we discussed the coded interviews based on our experiences from planning and conducting the workshops and reflected on what went well and what we would change in the curriculum module. An overview of all workshops and content categorized by technology, pedagogy, content, and share can be found in Table 1. In the following, we present a description of each workshop before discussing the results from the interviews combined with our reflections in the next section labelled Lessons learned.

5.1 Workshop 1: Finding manipulatives

We started the workshop by introducing the learning objectives and invited the PSTs to reflect on those, their rationales for participating, and their expectations, first individually, then in pairs, and lastly, share their thoughts in an open discussion. We continued with a short presentation of representations in mathematics and presented manipulatives as one type of representation. We discussed pedagogical aspects that teachers should be aware of when working with manipulatives in the classroom, such as manipulatives need to be used long-term to be effective, and mathematical concepts can be taught starting with tangible manipulatives and gradually increasing the abstraction level to symbols. We concluded the presentation with an example task. *What object does not fit?* Using a set of 3D printed solids, PSTs should discuss which solid should be removed based on geometrical characteristics. Next, we introduced DF tools and techniques and Thingiverse as one online repository for DF resources. They were tasked to explore Thingiverse and find a manipulative they could use in the classroom in two groups. We did not specify a mathematical concept or school context to allow an open exploration of the tool. We provided the groups with two 3D printers, one laser cutter and one vinyl cutter, and tutorials on using them. All of them chose to use a 3D printer. We assisted the groups during the making of their manipulative. Finally, we gave the groups a pedagogical task to discuss how and why the

Table 1: Overview of workshop content categorized by technology, pedagogy, content, and share

	Technology	Pedagogy	Content	Share
Find manipulatives Workshop 1	Thingiverse Prusa Slicer 3D printer	Representations of mathematical concepts Four principles for working with manipulatives [27]	Geometrical concepts and properties *What object does not fit?*	Presentation of produced manipulative and planned classroom activity.
Adapt Workshop 2	Inkscape Laser cutter Cricut Design Space Vinyl Cutter	Low threshold-high sealing tasks.	Fractions MatteList [32]	Tips for 3D printing Presentation of produced manipulative and planned classroom activity
Workshop 3	Tinkercad Prusa Slicer 3D Printer	Four stage framework for using manipulatives in teaching [16]	Parabola *Explore the relation between manipulative and quadratic functions.*	Tips for laser cutting
Create Workshop 4	Design Thinking Thingiverse Remix	Curriculum: different representations and core elements	Pyramid Algebra, Probability	Roleplay of PSTs teaching activity Presentation of planned manipulative and classroom activity Share on Thingiverse Feedback on workshops

manipulative is appropriate to use in the classroom. At the end of the workshop, the groups presented their produced manipulative, including their rationale for choosing it, how they would integrate the manipulative in a classroom activity, and their experiences of using Thingiverse for finding manipulatives.

5.2 Workshop 2: Adapting manipulatives

The second workshop started with a reflection task on the previous workshop. PSTs were asked to reflect on what went well, problems that occurred, and their thoughts on using 3D printing for making manipulatives. We asked them to provide tips for using the 3D printer. The PSTs worked in the same groups as in the first workshop and shared their tips on our local learning platform. Finally, PSTs were asked to produce the same manipulative from the previous workshop using each other's tips for 3D printing. We continued with presenting problem-solving tasks with low threshold and high sealing, exemplified by a classroom activity on fractions retrieved from a national mathematical resource page [32]. We created fractional bars for this exercise using a laser cutter and a vinyl cutter. We chose laser cutting as DF technology in the second workshop. We presented the essential functionality of a laser cutter and vinyl cutter and provided a tutorial to Inkscape and Cricut Design Space for creating 2D models. The groups were asked to produce the manipulative from the first workshop using a laser cutter and vinyl cutter. They had to reflect on how they would adapt the manipulative to a 2D model and if this changed the classroom activity. We asked them to sketch the 2D model on paper, adding measurements before creating the digital model in Inkscape. We assisted them using the laser cutter. At the end of the workshop, the groups presented their manipulative and made changes. They also shared their thoughts on using a laser cutter to make manipulatives for mathematics education.

5.3 Workshop 3: Adapting manipulatives

The third workshop started with a group discussion about the created manipulatives from a technological, pedagogical, and content perspective. Then, we asked them to share tips for using the laser cutter. None of the groups had used a vinyl cutter in the previous workshop, so we demonstrated the vinyl cutter at the end of the reflection round. We presented a four-stage framework [16] for using manipulatives in the classroom. As an example task, we produced a manipulative reifying a quadratic function. The PSTs task was twofold; first, to solve mathematical problems, e.g., how, and why the parabola is moving when changing the b-coefficient in a quadratic function $y = ax^2 + bx + c$, second, reflect on how to improve the manipulative to reify better the mathematical concept of a graph representing the quadratic function. As DF technology, we focused on 3D modeling in the third workshop. We introduced Tinkercad as modeling software to modify or create manipulative for 3D printing. The PSTs followed the built-in tutorial to learn basic functionality in Tinkercad. For the next task, PSTs worked in groups again. They could select from several manipulatives available on a table (including their previously produced manipulatives and some new ones). The task was to discuss how the manipulative should be adapted to reify the mathematical concept better and design a suitable teaching activity. The groups presented what manipulative they chose, how they wanted to adapt it, and how they planned to integrate it into a teaching activity. The remaining time they worked on their manipulative in Tinkercad. The PSTs did not finish their manipulative in this workshop, and we decided to continue the work in the next workshop.

5.4 Workshop 4: Creating manipulatives

The fourth workshop started with a reflection task about the manipulative they started modeling during the third workshop. The PSTs were asked to describe how the manipulative will be used for teaching in the learning platform and how the proposed changes improve the manipulative. The groups had time to finish their manipulative and perform a teaching activity as roleplay, where we researchers acted as pupils. In the second part of the workshop, we presented a design thinking process for creating new manipulatives proposed by Hjorth et al. [9] and how they can share their new manipulatives as remixes or novel artifacts in Thingiverse. For example, we presented a created manipulative reifying the relation between the volume of the pyramid and cube. We had prepared a task for a short design process for the PSTs. They chose a competence goal for algebra or probability from the 5^{th}-10^{th} grade mathematics curriculum as a point of departure and followed the design thinking process to prototype a new manipulative and a teaching activity for that competence goal. However, we did not have the time to implement the task during this workshop. The workshop concluded with a reflection task on PSTs' experiences from the workshops, their thoughts on how DF can be integrated into mathematics teacher education, and their suggestions for improving the conducted workshop series.

6 LESSONS LEARNED

We discuss lessons learned grouped by the learning objectives and workshop designs.

6.1 Learning objective: Students can produce manipulatives for mathematics teaching using DF technologies

Learning DF technologies was PSTs primary motivation for joining the project. During the workshops, we introduced several DF tools and techniques: Thingiverse to find and share digital resources; Tinkercad for modeling 3D designs; PrusaSlicer and Prusa printer for printing 3D artifacts; Inkscape and Cricut Design Space to model 2D designs; as well as a laser cutter and a vinyl cutter. In the workshops, PSTs got hands-on experience with all DF tools and techniques. All PSTs created 3-4 different manipulatives in the workshops, reifying fractions, angles, Pythagoras theorem, and binominal theorem (see Figure 3). Although PSTs report that they liked to learn many different DF skills and feel confident using the introduced tools and techniques, we found that most resources on Thingiverse were designed for 3D printing and that PTSs chose 3D printing as their first choice. One PST points out that "it is just essential to have the skills actually to make something or use a 3D printer. Many schools have 3D printers now." These findings align with Stigberg's [26] review reporting that 3D printing is the most common technology for creating manipulatives reifying mathematical concepts in previous research. However, we see the relevance of introducing several DF technologies to afford flexibility in choosing the appropriate tool for making manipulatives. For example, PSTs report that making 2D models for a laser cutter and vinyl cutter was easier than 3D modeling. Furthermore, they mention the benefits of using the laser cutter to make larger manipulatives with shorter production times. So far, laser cutter technologies are expensive, but one could argue that they will become more commonplace in schools like 3D printers today.

6.2 Learning objective: Students have critical thinking and problem-solving skills for designing manipulatives and classroom activities.

PSTs report that the combination of technology, pedagogy, mathematical content, and sharing activities in the workshops helped them understand the complexity of manipulatives for teaching. One PST highlights "how concrete you should be, when you put together a teaching plan with the help of manipulatives, that you should not look at the task lightly and think that you can solve it along the way." Another PST described that he learned that "it is not only to bring these [manipulatives] because they are nice figures to show. But actually, planning how to introduce them and what to do with them. And the fact that a manipulative is not a teaching plan, but it is an aid for your teaching plan." A third PST describes the process of making manipulatives for teaching in the following: "When we started looking at Thingiverse and looking at all the different things, I really wanted to use it all, and then I actually had to go in the curriculum and see what students should learn and found out that it might not be that relevant after all, although I would like to use it." We can see that both making a manipulative and classroom activity, the need to present and concretize ideas and rationales, and our focus on reflective activities in the workshops support PSTs' critical thinking and problem-solving skills. A PST sums it up as "I think it actually turned out very well, because like that. . . Often, I sat there and. . . disappeared into the technology. I learned a lot of new things in the technological parts. But then you reminded us that this is to be used for manipulatives and got us into the didactics and the fact that we should somehow produce something that can be used in teaching. So, I think it was a good distribution." PSTs pointed out the importance of sharing activities for receiving feedback and starting discussions between themselves and lecturers.

6.3 Learning objective: Students can collaborate in making manipulatives and are able to communicate their ideas and rationales to others

We can see four levels of sharing affording collaboration and communication in the workshops. (1) PSTs worked in groups making manipulatives and classroom activities. Sharing ideas and experiences and negotiating decisions were necessary for this collaboration. We could see that one group worked together throughout the entire project, whereas the other group divided tasks and decided to work individually. (2) Groups sharing their results and reflections during the workshops helped them communicate and concretize their ideas and get feedback from each other and lecturers. (3) In two workshops, we asked PSTs to provide tips for 3D printing and laser cutting on an online learning platform as a third way to share their experiences. (4) Finally, in the last workshop, we asked PSTs

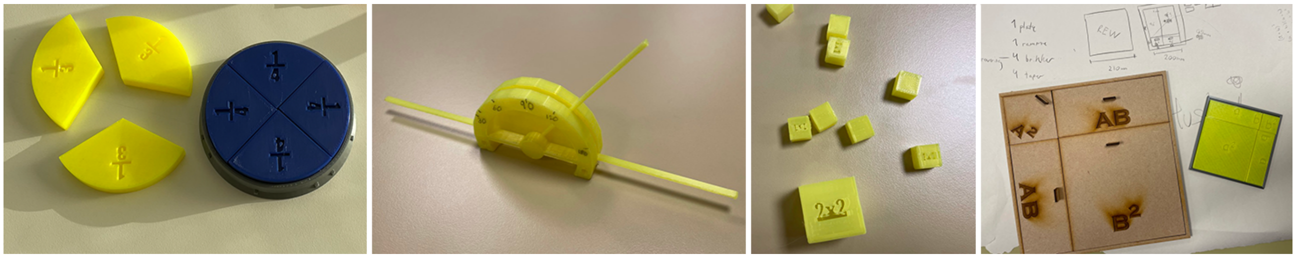

Figure 3: Examples of manipulatives produced by PSTs

to share their manipulatives on Thingiverse for everyone to use. Unfortunately, none of the groups shared their manipulative to Thingiverse. Sharing within the group, the workshop, and the online learning platform was experienced as unproblematic. However, PSTs expressed concerns when asked to share with Thingiverse. One PST describes this issue: "No, I have been a bit selfish. I make something because I want to make it, and there is no one who needs what I have made because it is only for myself. So, I don't feel I have become part of a community yet." Others are positive about sharing with others but want to ensure that the models are good enough. "I have not shared it yet, but I do not mind sharing it. . . The one I produced, I could have posted it, but then it's there to see. . . Now we produced it in the smallest possible size. So, if I had gotten it bigger and seen how it actually worked, I would not mind posting it." Another PST agrees that "Yes, I think I'll upload it. I made a small volume thing, so I think it's perfectly fine to upload because someone may want to use it. Who knows?" We see the importance of including different sharing activities from the workshops, and we will stress more sharing activities in the revised DF curriculum module.

6.4 Find-Adapt-Create-Share Framework

We proposed the find-adapt-create-share framework to scaffold making manipulatives for mathematics education using DF technologies. None of the PSTs had previous DF skills, so we decided to start with finding manipulatives. We continued with two workshops where PSTs adapted manipulatives to acquire more DF skills with different levels of sophistication. In the second workshop, PSTs adapted printing settings to improve manipulatives from the first workshop, and they copied previously selected 3D manipulatives to 2D models for laser cutting. In the third workshop, PSTs adapted 3D models in Tinkercad, which was more time-consuming than expected, and we decided to continue with the task in the fourth workshop. In the fourth workshop, we did not finish the planned design project to create manipulatives and apply previous DF skills. We will add a fifth workshop in a revised DF curriculum module focusing on creating manipulatives. In retrospect, we experienced the fourth workshop as a transition between adapting and creating manipulatives; one group created a new manipulative for the chosen mathematical concept. In contrast, the other group created a new component for an existing manipulative. The PSTs report that the progression in the workshops gave them a sense of achievement, offering low threshold and high ceiling tasks. "I think it has been a good progression then we first started to find something, and then we should make some improvements then and so." Another PST expresses, "It was a challenge enough, but at the same time. . . You can see it a bit like that there is a very low entry threshold to be able to produce or so. Make something of your own then, it does not have to be so advanced, but if you want to challenge yourself, you can. Yes make it difficult then." Another PSTs agrees, "I think that has been good. It was a low entry threshold and high ceiling. If you do not know anything about it, then it is still easy to get started, and you get done something, and if you know something, then you can further develop it with limitations on what you yourself want then." In the following, we will briefly discuss both the find and adapt level of the framework.

6.4.1 Find manipulatives. We started with finding manipulatives in Thingiverse. We chose to have an open task and asked PSTs to find any manipulative. That approach received split feedback from the PSTs. PSTs describe that "I think, if you have not done any of this before, then it is very cool to find something to print and see that it actually becomes a figure." and "Yes, I would probably rather go and look at Thingiverse first because I'm a little more like that I need a little inspiration and something. Em, I think it was hard to come up with something like that out of the blue." On the other hand, one PST felt, "If you had a goal to work from, then you would perhaps be a little more aware of what you should actually do or find to actually reach that goal here." There are both advantages and disadvantages to both approaches. A problem-oriented task would probably result in a more focused search, but we imagine that PSTs would miss out on exploring Thingiverse and the variety of available manipulatives. One PST sums it up: "Some will need a little inspiration and actually need to see a little what can be done and what already exists, while others can just jump right into it and invent something just like that. So, it is, in a way. . . It is not necessary to go to Thingiverse first, but I would at least think that it can be a good help for many who may not be able to fully imagine what to do right away."

PSTs see the benefits of using Thingiverse to get an overview of available resources and inspiration from others. "Even though you may want to do your own thing, I would still say that it is good to see what others have done before, and see a little where. . . Both where the limitation is and what the opportunity is." The predominant language in the Thingiverse community is English, and PSTs have stressed the need to have an overview of the English terms of mathematical concepts. "Because it's in English, so we cannot search for konkreter [Norwegian for manipulative]. What is it called in English?" or "Useful but at the same time, for me, it was a bit difficult to search for manipulatives. So, you should almost have some keywords first for what we can search for." Including an activity to find English terms for mathematical concepts is helpful in a revised DF curriculum module.

The second part of the first workshop aimed to produce manipulatives. Again, both groups chose 3D printing as DF technology. "I liked best to make the manipulative on a 3D printer. I think it was very cool, and I got a feeling of mastery from it. And that again then, when you sort of create something of your own, you kind of get a sense of belonging to it." Another PST agrees: "The advantage is that it gets very fast with the 3D printing. If you are a little restless, then I would say that you start immediately with the 3D print, then it becomes a bit like that. . . You get something in your hands right away. So, I see it as an advantage." Overall, PSTs experienced the first workshop as a good entry-level to DF for making manipulatives.

6.4.2 Adapt manipulatives. PSTs adapted manipulatives with different levels of sophistication in workshops two to four. They explored different DF tools and techniques for 3D modeling and printing, as well as 2D modeling for a laser cutter and vinyl cutter. We can see that PSTs preferred 3D modeling and printing. Tinkercad as a 3D modeling tool was very appreciated. PSTs continued modeling manipulatives at home and suggested getting homework during concluding interviews. "I chose Tinkercad because you can edit it in a simple way with those objects or build something all by yourself. You can get very nice things even if you do not build something yourself, and you can change small things." Another PST explained: "So I think the more basic it is, the easier it is to get started with, and it's still a lot. . . There's still a very high ceiling in Tinkercad. . . Yes, there's a lot you can do that you may not see at the very beginning. And so, it's like that, but you can put them [solids] together in a lot of different ways, and you can make holes and so on." PSTs asked to explore programming in Tinkercad as well. "I might have tried to get it in by using programming in it. That you got time to actually try that bit, too, because it's a very useful thing if you are going to do 3D printing in general. Yes, and the fact that we did not get to visit it and learn a little about it, and I think it was a bit of a shame, so I will definitely get into it somehow." In a revised DF curriculum module, we would add a fifth workshop to provide PSTs with more time to explore the DF tools for adapting and creating manipulatives. We can see that PSTs iterate between adapting, creating, and evaluating manipulatives in Tinkercad common in a design process.

7 CONCLUSION

This article describes our efforts to design a DF curriculum module for mathematics teacher education. The design objective was to introduce DF techniques joint with mathematical concepts and the design of classroom activities to develop PSTs' technological, pedagogical, and content knowledge. Previous research reporting from professional development projects for mathematics teachers focuses on STEM and how teachers can integrate DF and design thinking in the classroom. Instead, we were interested in introducing DF to PSTs to aid them in creating manipulatives for mathematics teaching as part of their professional development. We present a "find-adapt-create-share" framework, including four meaningful activities for making manipulatives using DF. We implemented the proposed framework in four workshops and explored how DF can be introduced to PSTs. Following the framework provided a low entry threshold to DF and good progression. Even though PSTs expressed that learning DF technologies was their primary motivation for joining the project, they reflect that the combination of technology, pedagogy, mathematical content, and sharing activities in the workshops helped them understand the complexity of manipulatives for teaching. Finally, PSTs experience sharing as crucial for receiving feedback and discussing between themselves and lecturers. However, they are hesitant to share with the larger maker community. We will continue with our research efforts to revise the DF curriculum module and plan to assess the proposed framework in a formal teaching context. We are also interested in exploring how the framework can guide professional development of in-service teachers.

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